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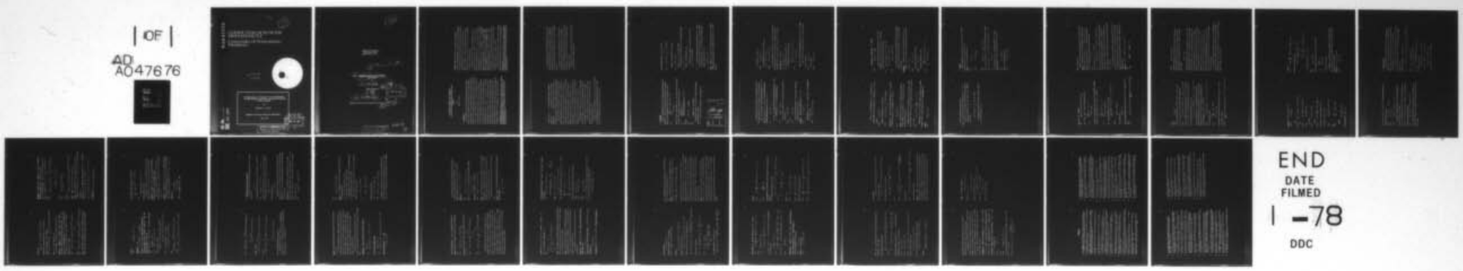

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ON THE ROLES OF "STABILITY" AND "CONVERGENCE" IN SEMIDISCRETE PROJECTION METHODS FOR INITIAL-VALUE PROBLEMS

by

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ON THE ROLE OF "STABILITY" AND "CONVERGENCE"* IN SEMIDISCRETE PROJECTIONS METHODS FOR INITIAL-VALUE PROBLEMS

by

Seymour V. Parter

1. Introduction

In the early 1950's as scientists became concerned with the numerical solution of partial differential equations there were many papers concerned with the questions of "stability" and "convergence" of solutions of difference approximations to time dependent problems (now called "equations of evolution").* In 1951 M. A. Hyman, S. Kaplan and G. G. O'Brien [34] discussed this question and described the Von Neumann "stability criterion". In the same year W. Leutert [32] gave an example of an "unstable" scheme which, nevertheless, was in some sense "convergent". These results were followed by many, many convergence proofs. (See [12], [24], [25], [26] for a few). In 1956 there appeared a paper by Jim Douglas Jr. [11] "On the Relation Between Stability and Convergence in the Numerical Solution of Linear Parabolic and Hyperbolic Differential Equations." However, Douglas was distracted by Leutert's example and restricted his efforts to the proof that stability (under appropriate "consistency" conditions) led to convergence. In the same year (1956) the

* No attempt at describing the history of this subject could possibly be complete. I make no claim that the above discussion is a complete description of the early pioneering papers. At the same time, no discussion of this topic can even begin without mentioning the fundamental paper by R. Courant, K. O. Friedrichs and H. Lewy [9].

fundamental paper of P. Lax and R. D. Richtmyer [30] employed the principle of uniform boundedness to show that if one demanded convergence for a sufficiently broad class of problems, then stability and convergence are indeed equivalent. This result is the famous "Lax Equivalence Theorem." In 1958 H. F. Trotter* [43] returned to the questions raised by Lax and Richtmyer and put the results (and theory) into the framework of the theory of Linear Semi-groups.

During this time an effort was made to understand and clarify the several possible definitions of "stability." In particular, in 1960 Strang [40] discussed "weak stability" in which the solution operator becomes unbounded as $\Delta t \to 0$ but at a rate which is $O(\Delta t^{-p})$. He proved the following beautiful theorem: <u>If the solution $u(x,t)$ is sufficiently smooth, then the discrete solution $U(x,t,\Delta t)$ of such a weakly stable method is convergent to $u(x,t)$ and the "rate of convergence" is that predicted by the truncation error.</u> In 1962 H. O. Kreiss [29] wrote a definitive paper on the relationship between various notions of stability, the Von Neumann Criterion and the concept of "Properly Posed in the Sense of Petrowsky" (see Aronson [1], Wendroff [44] also).

But here we are, some twenty years later, and most research in numerical methods for partial differential equations is not concerned with difference

* This famous paper is particularly interesting. Most numerical analysts don't realize that it is primarily devoted to the stability-convergence question, and, most probabilists, who-if they have read the paper-must know, seldom (if ever) mention this fact.

methods. The in[illegible] on Ritz-Galerkin methods, collocation methods, and [illegible] rojection Methods." And, as one reads the present day l[illegible] one rarely sees the word "stability." There are many, many "convergence" theorems (with appropriate smoothness assumptions).

Of course, there is a good reason for this state of affairs. Most Ritz-Galerkin methods with a continuous time variable are automatically stable. In fact, this observation is the beginning and the motivation for the paper by B. Swartz and B. Wendroff [42]- one of the early "American" papers on the subject of Galerkin methods for time dependent problems. Moreover, much of the research of today is concerned with a host of immediate questions- e. g. time discretization by multistep methods (see [2], [6], [10], [50] for a few), replacement of integration by quadrature methods (see [17] ,[37]), collocation (see [8], [15], [46]).

Nevertheless, particularly as we begin to look at more sophisticated projection methods, e. g. collocation, it seems reasonable to look again at this concept of "stability" and its relationship to "convergence."

In section 2 we formulate the problem of equations of evolution and semidiscrete numerical methods based on a sequence of subspaces $\{X_n\}$ and related projection operators $\{P_n\}$.

In section 3 we discuss some examples. In section 4 we use a modification of a now standard proof of the "Trotter Approximation Theorem" to discuss the roles of stability and convergence in a general setting.

This discussion explicitly shows how the semigroup theory clarifies much of the existing literature. In this connection, it is appropriate to mention that Helfrich [21] and Fujita and Mizutani [16] make explicit use of the theory of holomorphic semigroups in their treatment of parabolic problems.

In section 5 we discuss a particular definition of "weak stability" and show how one may obtain "convergence theorems" with such methods provided one has some additional smoothness and makes a particular choice of "initial values." The results of this section may be regarded as analogs of the theorem of Strang.

These results of section 5 are also closely related to results of Beals [3] for the partial differential equation.

2. The Problem

Let X be a Banach Space and let A be a densely defined linear operator from $D(A) \subset X$ into X. We are concerned with "semidiscrete" numerical methods for the approximate solution of the initial-value problem

$$\text{2.1)} \quad \begin{cases} \dfrac{d\,u(t)}{dt} = A\,u(t) + f(t)\,, & t > 0 \\ u(0) = u_0 \in X \end{cases}$$

where $f(t)$ is an X valued function of t. By a solution (see [22] page 619], [35, page 105] we mean an X-valued function $u(t)$ which is

(i) <u>continuous</u> <u>for</u> $t > 0$,

(ii) <u>continuously</u> <u>differentiable</u> (in t) for $t > 0$;

moreover,

(iii) for $t > 0$, $u(t) \in D(A)$,

and

(iv) equations (2.1) are satisfied.

We assume that equations (2.1) describe a "properly posed problem." To be more precise, we assume:

H.1 : A is the infinitesimal generator of a C_0 semigroup $T(t)$,

and, the unique solution of (2.1) is given by

$$\text{2.2)} \quad u(t) = T(t)\,u_0 + \int_0^t T(t-s)\, f(s)ds.$$

Moreover, the semigroup, $T(t)$, satisfies

$$\text{2.3)} \quad \|T(t)\| \le Me^{\omega t}$$

where $M > 0$, $\omega \ge 0$ are fixed constants.

A related problem is the "steady state" or time independent problem

$$\text{2.4)} \quad A\,u + f_0 = 0,$$

where f_0 is a fixed element of X .

We assume that this problem has a unique solution u for all $f_0 \in X$. In fact, we assume:

H.2: A^{-1} exists as a bounded linear operator defined on all of X . Moreover, the "resolvent condition" is satisfied, i.e., there is a constant M such that, for all real $\lambda > 0$, $(A-\lambda I)^{-1}$ exists as a bounded linear operator defined on all of X and

$$\text{2.5)} \quad \|(A-\lambda I)^{-n}\| \le M\lambda^{-n}$$

<u>Remark:</u> Assumption H.1 implies an estimate of the form of (2.5). Conversely, under appropriate assumptions on $f(t)$, assumption H.2 implies H.1. See [35, page 21].

A large class of numerical methods for the approximate solution of the steady state problem (2.4) are described in the following manner.

Let $\{X_n\}_1^\infty$ be a family of finite dimensional subspaces of X. (For convenience, let $\dim X_n = n$). Let $\{P_n\}_1^\infty$ be an associated family of uniformly bounded projections of X onto X_n with

2.6) $\|P_n\| \le M_0$.

Let $\{A_n\}_1^\infty$ be an associated family of nonsingular maps from X_n onto X_n . The approximant $u_n \in X_n$ satisfies the equation

2.7) $A_n u_n = -P_n f_0$.

In fact, the Galerkin method (or , the direct projection method) is obtained when

2.8a) $X_n \subset \mathcal{D}(A)$,

and

2.8b) $A_n = P_n A$.

A typical theorem associated with the above type of approximation scheme takes the following form.

Theorem T : There exists a Banach space $\tilde{X} \subset X$ with

2.9a) $\|y\| \le C \|y\|_{\tilde{X}}$

and a function $F(n) \downarrow 0$ as $n \to \infty$ such that: if u, the solution of (2.4), is in $\tilde{X}$ then

2.9b) $\|u - u_n\| \le F(n) \|u\|_{\tilde{X}}$.

If we define Q_n on $\mathcal{D}(A)$ by

2.10a) $Q_n = A_n^{-1} P_n A$

we can restate Theorem T (T for "typical") as:

Let $u \in \mathcal{D}(A) \cap \tilde{X}$; then

2.10b) $\|Q_n u - u\| \le F(n) \|u\|_{\tilde{X}}$.

Once one has developed this procedure for the steady state problem (1.4) and obtained Theorem T, one is naturally led to consider the following "continuous time, semidiscrete numerical method" for (2.1): Find a X_n-valued function $u_n(t)$ which is

(i) continuous for $t > 0$

(ii) continuously differentiable for $t > 0$

and satisfies the initial value problem

2.11) $$\begin{cases} \dfrac{d}{dt} u_n(t) = A_n u_n(t) + P_n f(t), & t > 0 \\ u_n(0) = U_{0,n} \in X_n \end{cases}$$

where $U_{0,n}$ is chosen in some prescribed way so that $\|u_0 - U_{0,n}\|$ is small.

In fact, there are two methods for choosing $U_{0,n}$ which come to mind at once. These are

2.12a) $U_{0,n} = P_n u_0$,

and, if $u_0 \in \mathcal{D}(A)$,

2.12b) $U_{0,n} = Q_n u_0$.

Since A_n is a linear map from X_n to X_n, and since X_n is of finite dimension, each A_n generates a C_0 semigroup $S_n(t) : X_n \to X_n$ given by

2.13a) $$S_n(t) = e^{A_n t} .$$

Moreover, the solution of (2.11) is given by

2.13b) $$u_n(t) = S_n(t)\, U_{0,n} + \int_0^t S_n(t-s)\, P_n f(s)\,ds.$$

Definition 2.1: The semi discrete method described by (2.11) is "stable" if there exist constants $\bar{M}$, $\bar{\omega}$ (independent of n) such that

2.14) $$\| S_n(t) \| \le \bar{M}\, e^{\bar{\omega} t} .$$

Remark: This definition of "stable" is classical and was introduced by Lax and Richtmyer [30] and Trotter [43]. The "norm" used in (2.14) is the norm of X restricted to X_n.

Applying the general theory of semigroups we find that the semi discrete method is stable if and only if there is a constant M_1 such that, for all real $\lambda > \bar{\omega}$ we have

2.15) $$\| (A_n - \lambda I)^{-m} \| \le \frac{M_1}{(\lambda - \bar{\omega})^m} , \quad n = 1, 2, \ldots .$$

Unfortunately, (2.15) is an infinite system of estimates and, in general, not easy to verify. A much stronger result is: the semigroups $S_n(t)$ satisfy

2.16a) $$\| S_n(t) \| \le e^{\bar{\omega} t}$$

if and only if for every real $\lambda > \bar{\omega}$ we have

2.16b) $$\| (A_n - \lambda I)^{-1} \| \le \frac{1}{(\lambda - \bar{\omega})} .$$

In many cases we find that the semigroup T(t) is not only a semigroup in X, but also is a semigroup in $\tilde{X}$. For this reason we will sometimes find it convenient to assume:

H.3: There are constants M_2 and α such that: if $x \in \tilde{X}$ then $T(t)x \in \tilde{X}$ and

2.17) $$\| T(t)x \|_{\tilde{X}} \le M_2 e^{\alpha t} \| x \|_{\tilde{X}}$$

We close this section with the observation that stable semidiscrete methods are "stable" under bounded perturbation. Specifically we have the following

Theorem 2.1: Suppose the semidiscrete method described by (2.11) is stable. Let $\{B_n\}$ be a family of uniformly bounded linear operators from X_n to X_n

2.18a) $$B_n : X_n \to X_n$$

and there is a constant B such that

2.18b) $$\| B_n \| \le B .$$

Consider the semidiscrete system

2.19) $$\begin{cases} \dfrac{dv_n}{dt} = (A_n + B_n)\, v_n + P_n f , \quad t > 0, \\ v_n(0) = V_{n,0} \in X_n , \end{cases}$$

Then this semidiscrete method is stable.

Proof: It suffices to consider the homogeneous case, i. e., $f \equiv 0$. Since (2.19) is a linear system of ordinary differential equations with constant coefficients, there is a solution $v_n(t)$. Moreover, we may write

$$v_n(t) = S_n(t)\, V_{n,0} + \int_0^t S_n(t-s) B_n\, v_n(s)\, d_s \; .$$

The theorem now follows from Gronwall's Inequality, see [4] and the basic estimate (2.14).

3. Examples

Before proceeding to the development of the general theory, we present some examples which are of particular interest.

Example 1: Let Ω be a smooth domain in R_n.

Let

3.1) $$X = L^2(\Omega) \; ,$$

3.2a) $$A = \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2} \; ,$$

and

3.2b) $$D(A) \equiv \{u \in X \; ; \; u \in \mathring{H}_1(\Omega) \cap H_2(\Omega)\}.$$

Let $X_n \subset D(A)$ be chosen so as to satisfy certain approximation properties (as in [2], [7], [14]). Let P_n denote L^2 projection onto X_n. Let

3.3) $$A_n = P_n A .$$

In this case we are dealing with Galerkin's Method for the classical Dirichlet problem. A typical result (Theorem T) takes the form:

Let

3.4a) $$\tilde{X} \equiv \{u \in D(A^{k_0})\},$$

where $k_0 > 1$ is an integer. Let

3.4b) $$\| u \|^2_{\tilde{X}} = \sum_{j=0}^{k_0} \| A^j u \|^2.$$

Then, **for** $u \in \tilde{X}$. **we have**

3.5) $$\|Q_n u - u\| = \|A_n^{-1} P_n A u - u\| \le F(n) \|u\|_{\tilde{X}} .$$

See [7], [21] ,[46]. Turning to the parabolic problem

3.6) $$\begin{cases} \frac{\partial u}{\partial t} = \Delta u = A u , \quad t > 0 \\ u(x,0) = u_0(x) \end{cases}$$

we see that it is relatively easy to show that the semidiscrete procedure is stable. In fact, we have: if $u_n(t) \in X_n$ satisfies

3.7n) $$\begin{cases} \frac{\partial u_n}{\partial t} = P_n \Delta u_n , \quad t > 0 \\ u_n(x,0) = U_{n,0}(x) \in X_n \end{cases}$$

then after multiplication by $u_n(t)$ we obtain

$$\frac{1}{2} \frac{d}{dt} \|u_n(t)\|^2 = (u_n, P_n \Delta u_n) = (u_n, \Delta u_n) \le 0.$$

Hence

$$\|u_n(t)\| \le \|U_{n,0}\|$$

which implies that

$$\|S_n(t)\| \le 1 .$$

Thus, one easily obtains results of the form: **if** $u(x,t)$ **and** $u_t(x,t) \in \tilde{X}$,
then

3.8) $$\|u(.,t) - u_n(.,t)\| \le C(t) F(n) \max_{0 \le s \le t} [\|u(.,S)\|_{\tilde{X}} + \|u_t(.,S)\|_{\tilde{X}}],$$

See [7] , [13] , [36].

Finally, in this case, the basic hypotheses H.1, H.2 and H.3 all hold.

Example 2: Choose X and A as in the previous example. However, we now require only that the subspaces $X_n \subset X$ belong to $H_1(\Omega)$. Let $(\ , \ )$ denote the inner product in $L^2(\Omega)$ and $< , >$ denote the inner product in $L^2(\partial\Omega)$. The numerical method for the steady state problem

3.9a) $$\Delta U + f$$

takes the form: find $u_n \in X_n$ such that

3.9b) $$(\nabla u_n, \nabla v_n) + (f, v_n) + n^{\sigma} < u_n, v_n > = 0 \quad \text{for all } v_n \in X_n .$$

Here σ is a positive constant. In this case we are dealing with the "penalty" method for the *Dirichlet problem*. The appropriate P_n is again the L^2 projection onto X_n. However, the operator A_n is a perturbation of the Galerkin operator. This problem has been analyzed under appropriate conditions on X_n , see [7].

Our next example is one of particular interest from the point of view of the questions raised in this report. Convergence theorems have been proven by J. Douglas and T. Dupont [15] and by J. H. Cerutti and S. V. Parter [8]. However, these authors have not touched on the questions of stability in the appropriate norm.

Example 3: Let $X = C[0,1] \cap \{ u ; u(0) = u(1) = 0 \}$

Let

3.10a) $$A = (\frac{d}{dx})^2 + a(x) \frac{d}{dx} + c(x)$$

where

3.10b) $\quad c(x) < 0$

and $a(x)$ is a smooth function. Let $0 = x_0 < x_1 < \dots < x_m = 1$ and let $I_j = [x_{j-1}, x_j]$. Let k be a fixed positive integer and let

3.11a) $\quad X_n \equiv \{u(x) \in X \cap C^1[0,1] \; ; \; u|_{I_j} \in P_{k+2}, \; j = 1, 2 \dots m\}$

where P_{k+2} denotes the polynomials of degree $< k+2$, i.e. of "order" $k+2$. Let $\xi_1, \dots \xi_k$ be the Gaussian points on $[0,1]$ (see [8] or [15] for a more complete discussion) and let

$$\xi_{js} = x_{j-1} + \xi_s(x_j - x_{j-1}), \quad j = 1, \dots m, \quad s = 1, \dots k$$

be the local Gauss points. The collocation method for the steady state problem studied by deBoor and Swartz [5] (their work is far more general, but this is the case of interest here) is described by the following procedure. Find $u_n \in X_n$ such that

3.11b) $\quad (A u_n)(\xi_{js}) = f(\xi_{js}) \quad j = 1, \dots m, \quad s = 1, \dots k.$

For the parabolic problem

3.12) $$\begin{cases} \frac{\partial u}{\partial t} = Au + f(x,t) \\ u(x,0) = u_0(x) \end{cases}$$

The collocation method takes the following form: find $u_n(x,t) \in X_n$ (for each fixed t) such that

3.13a) $$\frac{\partial u_n}{\partial t}(\xi_{js}, t) = (Au_n)(\xi_{js}) + f(\xi_{js}, t), \; j = 1, 2, \dots m, \; s = 1, \dots k,$$

3.13b) $\quad u_n(x,0) = U_{n,0}(x) \in X_n.$

Both Dupont and Douglas [15] and Cerutti and Parter [8] showed that one obtains the same kind of error estimates for the parabolic problem as deBoor and Swartz [5] obtained for the elliptic (steady state problem) when one used

$$U_{n,0}(x) = Q_n u_0.$$

Those results showed convergence in the maximum norm. Yet none of these authors established stability in the maximum norm. In terms of the discussion of this report, Dupont and Douglas established stability in the H_1 norm and used the imbedding of $H_1[0,1]$ in $C[0,1]$ to establish convergence in the presence of sufficient smoothness. On the other hand, Cerutti and Parter established a certain "resolvent estimate": which (i) came from the H_1 stability and (ii) could be interpreted as a form of weak stability and (iii) was good enough to allow the use of the Laplace transform <u>in the case of smooth solutions</u>. As far as this author knows, the question of "maximum norm stability" for this collocation scheme is still an open problem.

Our next example shows that the validity of Theorem T (for the steady state case) does not imply the stability or general convergence of the time dependent numerical method. In this example the operator A_n is a perturbation of the Galerkin operator. Moreover, in this case $\tilde{X} = X$.

<u>Example 4:</u> Let $X = L^2[0,\pi]$. Let

3.14a) $$A = (\frac{d}{dx})^2$$

with

3.14b) $$\mathcal{D}(A) = \{u \in H_2(0,\pi) ; u(0) = u(\pi) = 0\} .$$

Let

3.15a) $$X_n = \text{span} \{ \sin jx\}^n_{j=1} ,$$

3.15b) $$y_n = \text{span} \{\sin nx\}.$$

We write

3.16a) $$X_n = X_{n-1} \oplus Y_n ,$$

and let

3.16b) $$A_n = [(\frac{d}{dx})^2] \oplus [-(\frac{d}{dx})^2] .$$

Of course P_n is the L^2 projection onto X_n.

If $u \in \mathcal{D}(A)$ and $Au = f$ with

3.17a) $$f \sim \sum_{j=1}^{\infty} f_j \sin jx \in L^2(0,\pi) ,$$

then

3.17b) $$u = \sum_{j=1}^{\infty} (^{-f_j}/_{j^2}) \sin jx .$$

Clearly

$$P_n f = \sum_{s=1}^{n} f_j \sin jx$$

and the solution of $A_n u_n = P_n f$

is given by

3.18) $$u_n(x) = \sum_{j=1}^{n-1} (^{-f_j}/_{j^2}) \sin jx + \frac{f_n}{n^2} \sin nx .$$

We have the easy error estimate

$$\|u-u_n\|^2 = \frac{4\,|f_n|^2}{n^4} + \sum_{j=n+1}^{\infty} |f_j|^2/_{j^4}$$

$$\le 4 \sum_{j=n}^{\infty} \frac{|f_j|^2}{j^4} \le 4\|f\|^2 \sum_{j=n}^{\infty} \frac{1}{j^4} \to 0 .$$

On the other hand, let $u_0 = \sin nx$.

Then

3.19) $$S_n(t)u_0 = e^{n^2 t} u_0$$

Thus, the semidiscrete method for the initial value problem

3.20) $$\begin{cases} \frac{\partial u}{\partial t} = Au \\ u(x,0) = u_0 \end{cases}$$

is not stable in any norm !!

In example 4 we are dealing with a perturbation of Galerkin's method (see (2.8a),(2.8b)). In our next example we have a direct projection method which appears to be unstable.

<u>Example 5:</u> Let $0 < \nu < 1$ and let

3.21) $$A = \begin{bmatrix} \nu & 1 \\ 1 & \nu \end{bmatrix} .$$

Let $u(x,t) = [u_1(x,t)\ ,\ u_2(x,t)]^T$ and consider the mixed initial value-boundary value problem

3.22a) $u_t = A u_x$, $0 \le x \le 1$

3.22b) $u(x,0) = u_0(x)$

3.22c) $u_1(0,t) = u_1(1,t) = 0$, $t > 0$

In [20] Max D. Gunzburger considered the following semidiscrete Galerkin approach to this problem.

Let X_m be determined by using cubic B-splines on a uniform mesh with $u_1^m(0) = u_1^m(1) = 0$ and $u_2^m(0)$, $u_2^m(1)$ unspecified. The semi discrete equations are obtained by requiring

3.23) $(u_t^m - A u_x^m\ ,\ v^m) = 0$ for every $v^m \in X_n$.

In his interesting report [20] Gunzburger asserts that computational results indicate instability. He discusses the possible reasons for these difficulties.

4. The Basic Results

In this section we prove the general theorems which are essentially restatements of the Lax-Richtmyer-Trotter results in our present context. The main result is that for stable semi discrete numerical methods of the form described by (2.11) we can "lift" the results of Theorem T.

Our first result is a modification and interpretation of a basic identity which is usually used in the proof of the Trotter Approximation Theorem (see Pazy [35]).

Lemma 4.1: For every $x \in X$ we have $(t > 0)$

4.1) $$A_n^{-1}[P_n T(t) - S_n(t) P_n]A^{-1}x = \int_0^t S_n(t-s)\,[A_n^{-1} P_n - P_n A^{-1}]T(s)\,x\,ds.$$

Proof: Let $t > 0$ be fixed and let

$$G_n(s) = S_n(t-s)\,A_n^{-1} P_n T(s) A^{-1} x\ .$$

Then $G_n(s)$ is a differentiable function of s, $0 < s < t$.

Using the basic relations

$$T(t)Az = A\,T(t)z\ ,\quad \text{for } z \in D(A)$$

$$\frac{d}{dt} T(t)z = A\,T(t)z\ ,\quad \text{for } z \in X\ ,\ t > 0\ ,$$

$$A_n S_n(t) = S_n(t) A_n \quad \text{in } X_n$$

$$\frac{d}{dt} S_n(t)z = A_n S_n(t) z\ ,\ \text{for } z \in X_n\ ,$$

we find that

$$\frac{d}{ds} G_n(s) = S_n(t-s)\,[A_n^{-1} P_n - P_n A^{-1}]\,T(s)\,x .$$

Integrating this last relationship from 0 to t yields (4.1).

Lemma 4.2: For every $u \in D(A^2)$ we have

4.2) $$Q_n T(t)u - S_n(t)Q_n u = \int_0^t S_n(t-s)\,P_n[Q_n T(s)\,Au - T(s)\,Au]\,ds .$$

Proof: Let $x = A^2 u$ and apply (4.1).

For the moment, we restrict our attention to the case $f(t) \equiv 0$.

Theorem 4.1: Suppose $f(t) \equiv 0$. Suppose H.1 and H.2 hold. Suppose that Theorem T holds and the semidiscrete method is stable, i.e. (2.14) holds. Let $u(t)$ be the solution of (2.1). Let $u_n(t)$ be the solution of (2.11) with $U_{n,0}$ given by (2.12b). Let $u(t)$ and $Au(t) = \frac{d}{dt} u(t)$ belong to $D(A) \cap \tilde{X}$. Then

4.3a) $$\|Q_n u(t) - u_n(t)\| \le \bar{M} M_0 F(n) \int_0^t e^{\bar{\omega}(t-s)} \|Au(s)\|_{\tilde{X}}\, ds$$

and

4.3b) $$\|u(t) - u_n(t)\| \le F(n)[\|u(t)\|_{\tilde{X}} + \bar{M} M_0 \int_0^t e^{\bar{\omega}(t-s)} \|Au(s)\|_{\tilde{X}}\, ds] .$$

Proof: Apply lemma 4.2 and Theorem T (under the integral sign) to obtain (4.3a). Then (4.3b) follows from the triangle inequality and Theorem T.

Theorem 4.2 : Suppose $f(t) \equiv 0$. Suppose H.1, H.2 and H.3 hold and the semidiscrete method is stable. Suppose Theorem T holds.

Let $u(t)$ be the solution of (2.1) and $u_n(t)$ be the solution of (2.11) with $U_{0,n}$ given by (2.12b). Let u_0 and Au_0 belong to $D(A) \cap \tilde{X}$.

Then

4.4a) $$\|Q_n u(t) - u_n(t)\| \le \bar{M} M_0 M_1 F(n) \left[\int_0^t e^{\bar{\omega}(t-s)} e^{\alpha s} ds\right] \|Au_0\|_{\tilde{X}}$$

$$\le (\bar{M} M_0 M_1)\; C(t)\; F(n)\; \|Au_0\|_{\tilde{X}}$$

where

4.4b) $$C(t) = \left[\int_0^t e^{\bar{\omega}(t-s)}\, e^{\alpha s}\, ds\right] .$$

Moreover,

4.4c) $$\|u_n(t) - u(t)\| \le F(n)\,[M_1 e^{\alpha t} \|u_0\|_{\tilde{X}} + (\bar{M} M_0 M_1)\, C(t)\, \|Au_0\|_{\tilde{X}}] .$$

Proof: Apply H.3 to (4.3a) and (4.3b) in Theorem 4.1 .

Remark: Note the differences in the hypotheses of Theorems 4.1 and 4.2. In Theorem 4.2 we assume u_0 and $Au_0 \in D(A) \cap \tilde{X}$, ● which is the hypothesis of Theorem 4.1 .

Definition 4.1: The semidiscrete method described by (2.11) is "convergent" if: for all $u_0 \in X$ and all $\tau > 0$ we have

4.5a) $$\max\{\|S_n(t)\, U_{0,n} - T(t)u_0\| \;;\; 0 < t < \tau\} \to 0 \text{ as } n \to \infty ,$$

● hence by H.3 $u(t)$ and $Au(t)$ belong to $D(A) \cap \tilde{X}$,

whenever

4.5b) $$\|U_{0,n}-u_0\| \to 0 \quad \text{as} \quad n \to \infty .$$

Theorem 4.3: Let

4.6) $$V \equiv \{ u \in D(A) \cap \tilde{X} \; ; \; A u \in D(A) \cap \tilde{X} \} .$$

Suppose H.1, H.2, H.3 *hold*. *Suppose Theorem* T *holds and the semidiscrete method is stable*. *Suppose* V *is dense in* X . *Then, the semidiscrete method is convergent*.

Proof: Let $u_0 \in X$. Let $\{ v^{(k)} \}_{k=1}^{\infty}$ be a sequence satisfying

(i) For every k, $v^{(k)} \in V$

(ii) $\|v^{(k)} - u_0\| \to 0$ as $k \to \infty$.

Then for every k we have

$$\|S_n(t)U_{0,n} - T(t)u_0\| \le \|S_n(t)[U_{0,n} - Q_n v^{(k)}]\| + \|S_n(t)Q_n v^{(k)} - T(t)v^{(k)}\| + \|T(t)[v^{(k)} - u_0]\| .$$

Given $\epsilon > 0$ we may choose k_0 so large that

$$(1 + M + \bar{M} e^{\bar{\omega} \tau}) \|u_0 - v^{(k)}\| \le \epsilon/10 .$$

Then

4.7) $$\begin{cases} \| S_n(t)[U_{0,n} - Q_n v^{(k_0)}]\| \le \bar{M} e^{\bar{\omega} t} \| U_{0,n} - u_0 \| + \\ \bar{M} e^{\bar{\omega} t} \|u_0 - v^{(k_0)}\| + \bar{M} e^{\bar{\omega} t} \|v^{(k_0)} - Q_n v^{(k_0)}\| \end{cases}$$

Thus, employing Theorem 4.2 with k_0 fixed we have

$$\lim_{n \to \infty} \|S_n(t) Q_n v^{(k_0)} - T(t) v^{(k_0)}\| = 0$$

and

$$\limsup \|S_n(t)U_{0,n} - T(t)u_0\| \le \epsilon/10 .$$

Hence the Theorem is proven.

Employing the "Principle of Uniform Boundedness" in what is now a familiar argument (see [30], [38]) we obtain the converse result.

Theorem 4.4: *Suppose* H.1 *and* H.2 *hold*. *Suppose the semidiscrete method is convergent*. *Then the method is stable*.

Returning to the general case when $f(t) \not\equiv 0$ we recall that H.1 includes the assumption that (2.1) has a solution $u(t)$ and this solution is given by (2.2).

Theorem 4.5: *Assume that* H.1, H.2, H.3 *hold*. *Assume that Theorem* T *holds and the semidiscrete method is stable*. *Let* $u(t)$ *be the solution of* (2.1) *while* $u_n(t)$ *is the solution of* (2.11). *Let* $u_0 = U_{n,0} = 0$.

Suppose that

$$f(t) \in D(A) \cap \tilde{X}$$

$$A f(t) \in D(A) \cap \tilde{X} .$$

Let $C(t)$ *be given by* (4.4b). *Then*

$$4.8)\quad \| Q_n u(t) - u_n(t) \| \leq \int_0^t (\bar{M} M_0 M_1)\, F(n)\, C(t-s)\, \| A f(s) \|_{\tilde{X}}\, ds$$

$$+ \int_0^t \bar{M} e^{\bar{\omega}(t-s)} \| (Q_n - P_n) f(s) \|_{\tilde{X}}\, ds$$

Proof: We have

$$Q_n u(t) - u_n(t) = \int_0^t [Q_n T(t-s) - S_n(t-s) P_n] f(s)\, ds .$$

That is,

$$Q_n u(t) - u_n(t) = \int_0^t [Q_n T(t-s) - S_n(t-s) Q_n] f(s)\, ds \; +$$

$$\int_0^t S_n(t-s) [Q_n - P_n] f(s)\, ds .$$

Thus the theorem follows from Theorem 4.2 .

Of course, one can now go on to assume that f is approximated by functions $f^{(k)} \epsilon V$. In this way one obtains general convergence proofs similar to Theorem 4.3 for the general case.

5. Weak Stability and the Laplace Transform

In the finite difference case—where the approximate solution is defined only at times

$$t_k = k \Delta t$$

- one sometimes defines "weak stability" by the condition (see [19],[29])

$$5.1)\quad \| S_n(tk) \| \leq \bar{M} e^{\bar{\omega} t_k} (\Delta t)^{-P} \qquad k = 1, 2 \ldots$$

where P is a fixed positive number. In analogy to this one might consider in the semidiscrete case a definition of weak stability by the condition

$$\| S_n(t) \| \leq \bar{M} e^{\bar{\omega} t} n^P , \quad t > 0$$

where we remember that

$$n = \dim X_n .$$

Unfortunately, at this time we have not seen how to effectively study condition (5.1). Thus, for our purposes it is useful to work with the resolvent conditions (2.15), (2.16b) as the basis of stability and a corresponding concept of "weak stability".

Definition 5.1: The semidiscrete method described by (2.11) is "weakly stable" if there exists a function $M_1(\sigma) > 0$, and two constants $\bar{\omega}$, q such that: for all λ with $\mathrm{Re}\,\lambda > \bar{\omega}$ we have $(A_n - \lambda I)^{-1}$ exists as a linear map taking X_n onto X_n and

$$5.2)\quad \| (A_n - \lambda I)^{-1} \| \leq M_1(\mathrm{Re}\,\lambda)\, |\lambda - \bar{\omega}|^q$$

<u>Remark</u>: Stability implies weak stability because of the equivalent forms (2.15), (2.16).

Once one has introduced such a "Resolvent Condition" for stability or weak stability one naturally turns to the Laplace transform (see Hille-Phillips [22]) as a tool of analysis (see Strang and Fix [41], Cerutti and Parter [8] for applied examples). Unfortunately this approach seems to demand deeper results for the steady state problems. On the other hand, we are able to obtain "convergence theorems" for the time dependent problem in this weaker setting.

In particular we consider an extension of Theorem T to the case of systems. We shall sometimes require the validity of a theorem of the following form.

<u>Theorem S,N</u>: <u>Consider the steady state system of equations</u>

5.3) $$\begin{cases} A\phi_N = f_N \\ A\phi_m = \phi_{m+1} + f_m, \quad m = 0, 1, \ldots N-1 \end{cases}$$

<u>and the related discrete system</u>

5.4) $$\begin{cases} A_n\phi_N = P_n f_N \\ A_n\phi_m = P_n(\phi_{m+1} + f_m) = \phi_{m+1} + P_n f_m, \ m = 0, 1, \ldots N-1 \end{cases}$$

<u>There is a Banach space</u> $Y \subset \tilde{X}$ <u>with</u>

$$\|y\|_{\tilde{X}} \leq C_1 \|y\|_Y$$

and

5.5) $$\|\phi_m - \Phi_m\| \leq C_2 F(n) \|\Phi_m\|_Y$$

<u>Remark</u>: It is perhaps worth noting that the example 4 of section 3 has the following properties:

(i) if $n > 2N$ then Theorem S,N is valid

(ii) the semidiscrete method is <u>not</u> weakly stable.

Before proceeding with the technical details of the arguments to come, it is perhaps worthwhile to sketch our approach.

Let $u(t)$ and $u_n(t)$ be the solution of (2.1) and (2.11) respectively. Consider their Laplace Transforms

5.6) $$\hat{u}(s) = \int_0^\infty e^{-st} u(t)dt,$$
$$\hat{u}_n(s) = \int_0^\infty e^{-st} u_n(t)dt.$$

These functions then satisfy

5.7a) $$s\,\hat{u}(s) = A\hat{u} + \hat{f}(s) + \hat{u}_0$$

5.7b) $$s\,\hat{u}_n(s) = A_n\hat{u}_n + P_n\hat{f}(s) + \hat{U}_{0,n}.$$

If we imagine s fixed then (5.7a) is a steady state problem similar to (2.4) which is solvable by virtue of the resolvent condition (2.5). Moreover, (5.7b) is a discretization of this problem based on the same subspaces

X_n and the projections P_n. Thus, if an appropriate Theorem T(s) holds we would have an estimate of the form (for sufficiently smooth $\hat{u}(s)$)

5.8) $\quad \|\hat{u}(s) - \hat{u}_n(s)\| \leq C(s)\, F(n)\, \|\hat{u}(s)\|_{\tilde{X}}$.

Applying the well known integral inversion formula [48] we have, with $s = \gamma + i\sigma$

5.9a) $$u(t) - u_n(t) = \frac{1}{2\pi i} \int_{\gamma - i\infty}^{\gamma + i\infty} e^{st} \, [\hat{u}(s) - \hat{u}_n(s)] \, ds \quad ,$$

or

5.9b) $$\|u(t) - u_n(t)\| \leq \frac{e^{\gamma t}}{2\pi} \int_{\gamma - i\infty}^{\gamma + i\infty} |C(s)| \, F(n) \, \|\hat{u}(s)\|_{\tilde{X}} \, |ds|$$

Unfortunately one must worry about a few technical details. In particular, there is the question of the convergence of the integrals in (5.9a), (5.9b). At this point it is worth noting that this question is really very different in these two cases. The integral appearing in (5.9a) is the usual integral of complex variable theory - the Cauchy limit as the interval of integration tends to ∞. Moreover, the term $e^{i\sigma t}$ enables one to employ (directly or indirectly) the Riemann-Lebesgue Lemma to aid in this convergence. The integral appearing in (5.9b) is a Lebesgue integral and its absolute convergence is required.

As indicated above we require an appropriate convergence theorem for steady state problems depending on the parameter s. We shall assume that someone has proven a theorem of the following form.

Theorem T(γ): Let $\gamma \geq \bar{\omega}$. For $u \in D(A) \cap \tilde{X}$

let

5.10a) $\quad Q_n(\gamma)\, u = (\gamma I - A_n)^{-1} P_n (\gamma I - A)\, u$.

That is, $u_n = Q_n(\gamma)\, u$ is the solution of the discrete problem

$$(\gamma I - A_n)\, u_n = P_n' (\gamma I - A) u.$$

Then, there is a function $M_2(\gamma)$ such that

5.10b) $\quad \|Q_n(\gamma) u - u\| \leq M_2(\gamma)\, F(n)\, \|u\|_{\tilde{X}}$

Remark: One might consider asserting a Theorem T(s) which gives similar results when γ is replaced by a complex number, say $\gamma + i\sigma$. However, since most of these problems arise as problems with real coefficients it is more reasonable to restrict oneself in this way.

Finally we require one technical lemma concerning the inversion formula (5.9a).

Lemma 5.1: Let $v(t)$ satisfy the appropriate growth conditions so that its Laplace transform

$$\hat{v}(s) = \int_0^{\infty} e^{-st}\, v(t)\, dt$$

exists for $\mathrm{Re}\ s > \bar{\omega}$. Let k be an integer > 0 and let $\gamma > \bar{\omega}$.

Let $\tau > 0$ be fixed and suppose that $t < \tau$

Then

$$\int_{\gamma-i\infty}^{\gamma+i\infty} e^{st} \left[\frac{1}{s^k} \int_{\tau}^{\infty} e^{-s\sigma} v(\sigma)\,d\sigma \right] ds = 0 .$$

Proof: This result is an immediate consequence of the usual formal formulae for the Laplace transform and its inverse - see [48]. Intuitively it asserts that the future cannot affect the present.

Remark: The growth conditions required in this lemma are easily verified in the applications to follow. For example, if $v(t)$ satisfies an equation of the form (2.1) then the appropriate estimates follow from the representation formula (2.2) and estimates on $f(s)$.

Our first result is a special case in which the "mild instability" is truly very mild.

Theorem 5.1: Suppose H.1 and H.2 hold. Suppose the semi discrete method is weakly stable with

5.11) $\quad -1 < q < 0$.

Suppose Theorem $T(\gamma)$ holds, Theorem T holds, $u_0 \in D(A) \cap \tilde{X}$ and $U_{0,n} = Q_n u_0$.

Furthermore, if

$$v(t) = u(t) - u_0 ,$$

we suppose that $v(t)$, $v'(t)$, $v''(t) \in D(A) \cap \tilde{X}$ and satisfy the appropriate growth conditions so that Lemma 5.1 applies. Then, there is a constant K such that

5.12) $$\| u(t) - u_n(t) \| \le F(n) \| u_0 \|_{\tilde{X}} +$$

$$F(n) e^{(\bar{\omega}+1)t} K \Big(\max_{0 \le \sigma \le t} \| \frac{du}{dt} \|_{\tilde{X}} + \max_{0 \le \sigma \le t} \| \frac{d^2u}{dt^2} \|_{\tilde{X}} \Big)$$

Proof: Let $v_n = u_n - U_{n,0}$; then

$$u(t) - u_n(t) = v(t) - v_n(t) + (u_0 - Q_n u_0).$$

Thus, since Theorem T holds, we need only study $v(t) - v_n(t)$. We have

$$\frac{dv}{dt} = Av + f(t) + Au_0 , \qquad v(0) = 0$$

$$\frac{dv_n}{dt} = A_n v_n + P_n [f(t) + Au_0] , \quad v_n(0) = 0$$

Applying the Laplace transform to these equations we have

5.13a) $\quad s\hat{v} = A\hat{v} + \hat{g}$,

5.13b) $\quad s\hat{v}_n = A_n \hat{v}_n + P_n \hat{g}$,

where

5.13c) $\quad \hat{g}(s) = \hat{f}(s) + \frac{Au_0}{s}$.

Let $s = \gamma + ib$ where γ is a fixed constant greater than $\bar{\omega}$, say $\gamma = \bar{\omega} + 1$.

Let $W_n(b)$ be the solution of the discrete problem

$$\gamma W_n = A_n W_n + P_n \left[\hat{g} - ib\hat{v} \right] .$$

Since we also have

$$\gamma \hat{v} = A\hat{v} + [\hat{g} - ib\hat{v}] \; ,$$

we obtain from Theorem $T(\gamma)$ the estimate

5.14) $$\| W_n(b) - \hat{v} \| \le M_2(\gamma)\, F(n)\, \| \hat{v}(s) \|_{\tilde{X}}$$

Moreover,

$$s(W_n - \hat{v}_n) = A_n (W_n - \hat{v}_n) + P_n \, [\, ib(W_n - \hat{v})] \, .$$

Applying (2.6) and (5.2) we have

5.15) $$\| W_n(b) - \hat{v}_n(s) \| \le M_1(\gamma)\, M_0\, |b|\, \| W_n(b) - \hat{v} \| \; |s - \bar{\omega}|^q$$

Using (5.14), (5.15) and the triangle inequality we have

5.16) $$\| \hat{v}(s) - \hat{v}_n(s) \| \le \left[1 + \frac{M_1(\gamma) M_0 |b|}{(1 + |b|^2)^{-q/2}} \right] \| \hat{v}(s) \|_{\tilde{X}}\, F(n)\, M_2(\gamma) \, .$$

Since $v(0) = 0$ we obtain

5.17) $$\hat{v}(s) = \frac{1}{s^2} \frac{dv}{dt}(0) + \frac{1}{s^2} \int_0^\infty e^{-st} \left[\frac{d}{dt} \right]^2 v(t) \;\; dt \, .$$

Let $\nu \;\; = -q/2 > 0$ and let

$$K = \frac{M_2(\gamma)}{2\pi} \int_{-\infty}^{\infty} \left[1 + \frac{M_0 M_1(\gamma) |\sigma|}{(1+\sigma^2)^{\nu}} \right] \frac{d\sigma}{1+\sigma^2} \; .$$

Then, applying lemma 5.1 and (5.9b) we have

$$\| v(t) - v_n(t) \| \le K\, e^{(\bar{\omega}+1)t} \left[\| \frac{dv}{dt}(0) \|_{\tilde{X}} + \max_{0 \le \sigma \le 1} \| \frac{d^2}{dt^2} v(\sigma) \|_{\tilde{X}} \right] F(n) \, .$$

Thus, the theorem follows from the observation that time derivatives of $u(t)$ are the same as time derivatives of $v(t)$.

<u>Remark:</u> The error estimate (5.12) should be compared with (4.3b) of Theorem 4.1. If $f(t) \equiv 0$ the estimate (4.3b) depends only on $\| u(t) \|_{\tilde{X}}$ and $\| \frac{du}{dt} \|_{\tilde{X}}$ while (5.12) also includes a term $\| \frac{d^2u}{dt^2} \|_{\tilde{X}}$.

This last term is (apparently) not introduced because of the weak stability (as opposed to stability) but is rather due to the Laplace transform approach-see the remarks following (5.9a), (5.9b). However, the resolvent estimate with $-1 < q < 0$ is a very strong estimate - a sort of "weak holomorphic semigroup stability." Such an estimate should not be expected unless $T(t)$ is itself a holomorphic semigroup. This would occur if (2.1) were a parabolic problem. In fact, it is this estimate that was exploited by Cerutti and Parter [8].

<u>Remark:</u> While the results of section 4 also seem to be based on the choice of $U_{0,n} = Q_n u_0$, there is a significant difference between those results and the result above. The stability assumption of section 4 allows for an immediate result for any $U_{0,n}$ close to u_0. In the Theorem above we are definitely limited to a restricted choice of $U_{0,n}$. This aspect of the theory will be very clearly emphasized in the more general result which follows.

<u>Definition 5.2</u>: Suppose Theorem S,N holds. Suppose

5.18) $\phi_j = \frac{d^j u}{dt^j}(0) \in D(A) \cap \tilde{X} \cap Y, \quad j = 0, 1, \ldots N.$

Then

5.19) $$\begin{cases} A\,\phi_N = A\,\phi_N \equiv f_N \\ A\,\phi_m = \phi_{m+1} - (\frac{d}{dt})^m f(0), \quad m = 0, 1, \ldots N-1 \end{cases}$$

Let $\phi_j(n)$, $j=0, 1, \ldots N$ be the solution of the corresponding discrete system

5.20) $$\begin{cases} A_n \phi_N(n) = P_n f_N, \\ A_n \phi_m(n) = \phi_{m+1}(n) - P_n \left[(\frac{d}{dt})^m f(0) \right], \quad m=0,1,\ldots N-1. \end{cases}$$

Then we let $\tilde{Q}_n$ be the operator which maps $u \to \phi_0(n)$, i.e.

5.21) $\tilde{Q}_n u = \phi_0(n)$.

<u>Theorem 5.2</u>: <u>Suppose</u> H.1 <u>and</u> H.2 <u>hold</u>. <u>Suppose the semidiscrete method is weakly stable with</u> $q \geq 0$. <u>Suppose Theorem</u> T(γ) <u>holds and Theorem</u> S,N <u>holds with</u>

5.22) $N > q + 1$

<u>Suppose</u> (5.18) <u>holds and</u>

5.23) $U_{0,n} = \tilde{Q}_n u = \phi_o(n)$.

<u>Suppose</u>

5.24a) $\phi_j = \frac{d^j u}{dt^j}(0) \in D(A) \cap X \cap Y, \; j = 0, 1, \ldots N$

<u>and</u>

5.24b) $\frac{d^{N+1}u}{dt^{N+1}} \in \tilde{X}$

<u>and satisfies the necessary growth conditions so that lemma 5.1 applies.</u>

<u>Then, there is a constant K so that</u>

5.25) $$\begin{cases} \| u(t) - u_n(t) \| \leq K\, e^{(\bar{\omega}+1)t} \max_{0 \leq \sigma \leq t} \| (\frac{d}{dt})^{N+1} u(\sigma) \|_{\tilde{X}}\, F(n) \\ \quad + C_2 F(n) \sum_{j=0}^{N} \| \phi_j \|_Y \frac{t^j}{j!}, \end{cases}$$

<u>Proof:</u> Let

5.26a) $v(t) = u(t) - \sum_{j=0}^{N} \phi_j \frac{t^j}{j!}$,

5.26b) $v_n(t) = u_n(t) - \sum_{j=0}^{N} \phi_j(n) \frac{t^j}{j!}$

Then a direct calculation shows that

5.27a) $\frac{dv}{dt} = Av + t^N \psi(t)\;, \; (\frac{d}{dt})^j v(0)=0, \; j=0, 1, \ldots N$

5.27b) $\frac{dv_n}{dt} = A_n v_n + P_n(t^N \psi(t)), \; (\frac{d}{dt})^j v_n(0)=0, \; j=0, 1, \ldots N$

where $\phi(t)$ is determined from the Taylor series expansion of $f(t)$ and $u(t)$ and $A\phi_N$.

Since Theorem S,N holds we have

$$\text{5.28)}\quad \begin{cases} \|u(t) - u_n(t)\| \le \|v(t) - v_m(t)\| + \sum_{j=0}^{N} \frac{t^j}{j!} \|\phi_j(n) - \phi_j\| \\ \le \|v(t) - v_n(t)\| + C_2 F(n) \sum_{j=0}^{N} \|\phi_j\|_\gamma \frac{t^j}{j!} \end{cases}$$

Therefore we need only study $\|v - v_n\|$. Taking the Laplace transform of (5.27a), (5.27b) we have

$$\text{5.29a)}\quad s\hat{v} = A\hat{v} + \hat{G}$$

$$\text{5.29b)}\quad s\hat{v}_n = A_n\hat{v}_n + P_n\hat{G}$$

where $G(t) = t^N \psi(t)$. Let $s = \gamma + ib$ with $\gamma = \bar{\omega} + 1$.

Let $W_n(b)$ be the solution of

$$\gamma W_n = A_n W_n + P_n [\hat{G} - ib\hat{v}] .$$

Then, as in the proof of Theorem 5.1, we obtain

$$\|W_n(b) - \hat{v}(s)\| \le M_2(\gamma) F(n) \|\hat{v}(s)\|_{\underset{\sim}{X}}$$

Moreover

$$s(W_n - \hat{v}_n) = A_n(W_n - \hat{v}_n) + P_n [ib(W_n - \hat{v})]$$

and therefore

$$\|W_n(b) - \hat{v}_n(s)\| \le M_1(\gamma) M_0 |b| \|W_n(b) - \hat{v}(s)\| \; |s - \bar{\omega}|^q .$$

This gives the estimate

$$\|\hat{v}(s) - \hat{v}_n(s)\| \le [1 + M_1(\gamma) M_0 |b| (1 + |b|)^q] \|\hat{v}(s)\|_{\underset{\sim}{X}} .$$

The conditions on $v(t)$ imply that

$$\hat{v}(s) = \frac{1}{s^{N+1}} \int_0^\infty e^{-st} [(\frac{d}{dt})^{N+1} v(t)] \, dt .$$

Let

$$\text{5.30)}\quad K = \frac{1}{2\pi} \int_{-\infty}^{\infty} [1 + M_1(\gamma) M_0 |b| (1 + |b|)^q] \frac{db}{(1 + |b|^2)^{(N+1)/2}} .$$

Then we have

$$\text{5.31)}\quad \|v(t) - v_n(t)\| \le K e^{(\bar{\omega}+1)t} \max_{0 \le \sigma \le t} \|(\frac{d}{dt})^{N+1} v(\sigma)\|_{\underset{\sim}{X}} F(n)$$

which proves the Theorem.

Remark: Perhaps it seems very artificial to suggest such special initial values. However, such choices have already appeared in the literature. For example, in Cerutti and Parter [8] just this choice was made in order to assure the "superconvergence" at the knots. Working on the same problem Dupont and Douglas [15]

employed an even more complicated algorithm to obtain an appropriate initial value. See [14],[32] also.

In the general case where the semidiscrete method is only weakly stable one would not expect an analog of Theorem 2.1. Indeed, in [29] Kreiss gave an example that shows that in his theory such a result is impossible. However, when (5.11) holds, i.e. when we have "weak holomorphic semigroup stability," we may have such a result.

Theorem 5.3: Suppose the semigroups $S_n(t)$ satisfy (5.2) with $-1 < q < 0$. Let B_n be a family of linear operators satisfying (2.18a), (2.18b). Suppose also that $M_1(\sigma)$ "grows slowly," In particular, suppose there are real constants, $\bar{\gamma} > \bar{\omega}, \rho$, with $0 < \rho < 1$ such that: if

5.32a) $\quad \operatorname{Re} \lambda > \bar{\gamma}$

then

5.32b) $\quad M_1(\operatorname{Re}\lambda) |\lambda - \bar{\gamma}|^q \quad B < \rho$.

Then we have the modified resolvent estimate: For all λ with $\operatorname{Re} \lambda > \bar{\gamma}$ we have

5.33) $\quad \|(A_n + B_n - \lambda I)^{-1}\| \le \left[\dfrac{M_1(\operatorname{Re}\lambda)}{1-\rho}\right] |\lambda - \bar{\gamma}|^q$.

Proof: Consider the system

$$(A_n + B_n - \lambda I)\, u = f .$$

Then, if $\operatorname{Re} \lambda > \bar{\gamma}$,

$$u = (A_n - \lambda I)^{-1} [f - B_n u] .$$

Using (5.2) and (5.32b) we have

$$\|u\| \le M_1(\operatorname{Re}\lambda)\, |\lambda - \bar{\omega}|^q \, \|f\| + \rho \|u\| .$$

Since $\bar{\omega} \le \bar{\gamma}$ and $q < 0$, we have

$$|\lambda - \bar{\omega}|^q \le |\lambda - \bar{\gamma}|^q$$

and (5.33) follows at once.

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